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Duality in Vector Optimization in Banach Spaces with Generalized Convexity

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Abstract. We consider a vector optimization problem with functions defined on Banach spaces. A few sufficient optimality conditions are given and some results on duality are proved.

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1. Introduction

Let E, F and G be three Banach spaces. Consider the following mathematical programming problem:

$$\min\{f(x)\colon x\in C, -g(x)\in K\}, \tag{P}$$

where f and g are mappings from E into F and G, respectively, and C and K are two subsets of E and G.

This problem has been investigated extensively in recent years. When F and G are finite-dimensional linear spaces, and f and g are locally Lipschitz, problem (P) was studied by Clarke [1], Craven [2], Minami [3], Giorgi and Guerraggio [4], Reiland [5], Lee [6], Liu [7], Mishra and Mukherjee [8], Mishra [9], Kim [10] and Bhatia and Jain [11] among others.

The Lipschitz infinite dimensional case was considered by El Abdouni and Thibault [12], Coladas, Li and Wang [13] and recently, by Brandão, Rojas-Medar and Silva [14].

Brandão, Rojas-Medar and Silva [14] studied multiobjective mathematical programming with non-differentiable strongly compact Lipschitz functions defined on general Banach spaces.

Under a Slater-type condition and an invexity notion for mappings defined between Banach spaces, Karush-Kuhn-Tucker type conditions and Mond-Weir type duality results are established in [14].

In this paper, we extend the concept of type I functions [15], pseudotype I and quasitype I functions [16], quasipseudotype I, pseudoquasitype I [17] to the context of Banach spaces and establish the sufficiency of Karush-Kuhn-Tucker type optimality conditions under *weaker invexity* assumptions than that of Bandão, Rojas-Medar and Silva [14]. We also obtain various duality results under aforesaid assumptions.

This paper is organized as follows. In Section 2, we fix some basic notation and Terminology. In Section 3, we establish sufficient optimality conditions and finally in Section 4, we obtain various duality theorems.

2. Preliminaries and Definitions

The Clarke generalized directional derivative of a locally Lipschitz function from E into $\mathbb{R}$ at $\bar{x}$ in the direction d, denoted $\phi^0(\bar{x}; d)$ (see Ref. 1), is given by

$$\phi^0(\bar{x}; d) = \limsup_{\substack{x \to \bar{x} \\ t \downarrow 0}} [1/t][\phi(x+td) - \phi(x)].$$

The Clarke generalized gradient of ϕ at $\bar{x}$ is given by

$$\partial\phi(\bar{x}) = \{x^* \in E^*\colon \phi^0(\bar{x}, d) \geqslant \langle x^*, d\rangle,\ \forall d \in X\},$$

where E^* denotes the topological dual of E and $\langle\cdot,\cdot\rangle$ is the duality pairing.

Let C be a nonempty subset of E and consider its distance function, i.e., the function $\delta_C(.)\colon E \to \mathbb{R}$ defined by

$$\delta_C(x) = \inf\ \{\|x - c\|\colon c \in C\}.$$

The distance function is not everywhere differentiable, but is globally Lipschitz.

Let $\bar{x} \in C$. A vector $d \in E$ is said to be tangent to C at $\bar{x}$ if

$$\delta_C^0(\bar{x}, d) = 0.$$

The set of tangent vectors to C at $\bar{x}$ is a closed convex cone in E, called the (Clarke) tangent cone to C at $\bar{x}$ and denoted by $T_C(\bar{x})$.

DEFINITION 1 [14]. A mapping $h\colon E \to G$ is said to be strongly compact Lipschitzian at $\bar{x} \in E$ if there exist a multifunction $R\colon E \to \text{comp}(G)$[comp (G) = the set of all norm compact subsets of G] and a function $r\colon E \times E \to \mathbb{R}_+$ satisfying the following conditions:

(i) $\lim_{x \to \bar{x}, d \to 0} r(x, d) = 0$;

(ii) there exists $\alpha > 0$ such that
$t^{-1}[h(x+td)-h(x)] \in R(d)+\|d\|r(x,t)B_G$, for all $x \in \bar{x}+\alpha B_G$ and $t \in (0,\alpha)$, where B_G denotes the closed unit ball around the origin of G;

(iii) $R(0)=\{0\}$ and R is upper semicontinuous.

Remark 1. If G is finite-dimensional, then h is strongly compact Lipschitian at $\bar{x}$ if and only if it is locally Lipschitz near $\bar{x}$. If h is strongly compact Lipschitzian, then for all $u^* \in G^*$, $(u^* \circ f)(x) = \langle u^*, h(x) \rangle$ is locally Lipschitz. For more details about strongly compact Lipschitzian mapping, we refer readers to (Ref. [12]).

From now on, let $Q \subset F$ and $K \subset G$ denote pointed closed convex cones with nonempty interior; let Q^*, K^* be their respective dual cones. The cone Q induces a partial order $\leqslant$ on F given by

$$z' \leqslant z \quad \text{if } z - z' \in Q; \tag{1}$$

$$z' < z \quad \text{if } z - z' \in \operatorname{int} Q \tag{2}$$

$z' \not\leqslant z$ is the negation of (1) and $z' \not< z$ is the negation of (2). Analogously, K induces a partial order on G.

Phuong, Sach and Yen [18] introduced the following notion of *invexity* for a locally Lipschitz real-valued function ϕ: $E \to \mathbb{R}$, with respect to a nonempty set $C \subset E$.

DEFINITION 2. ϕ is said to be *invex* at $x \in C$, with respect to C, if for every $y \in C$, there is $\eta(y,x) \in T_C(x)$ such that

$$\phi(y) - \phi(x) \geqslant \phi^0(x; \eta(y,x)).$$

ϕ is invex on C if this inequality holds for every $x, y \in C$.

Following Phuong, Sach and Yen [18], Brandão, Rojas-Medar and Silva [14] extended the notion of invexity for functions between Banach spaces in a broad sense, as follows.

DEFINITION 3. f: $E \to F$ and g: $E \to G$ are *invex* if $u^* \circ f$ and $v^* \circ g$ are invex in the sense of Definition 2.2, for all $u^* \in Q^*$ and $v^* \in K^*$.

We extend the notions of type-I [15], pseudotype I, quasitype I [16] and pseudoquasitype-I and quasipseudotype-I [17] functions in the sense of Phuong, Sach and Yen [18].

DEFINITION 4. Locally Lipschitz real-valued functions f: $E \to \mathbb{R}$ and g: $E \to \mathbb{R}$ are said to be *type-I* at $x \in C$, with respect to C, if for every $y \in C$, there is $\eta(y,x) \in T_C(x)$ such that

$$f(y) - f(x) \geqslant f^0(x; \eta(y,x));$$

$$-g(x) \geqslant g^0(x; \eta(y,x)).$$

DEFINITION 5. (f,g) is said to be *quasitype-I* at $x \in C$, with respect to C, if for every $y \in C$, there is $\eta(y,x) \in T_C(x)$ such that

$$f(y) \leqslant f(x) \Rightarrow f^0(x;\eta(y,x)) \leqslant 0;$$
$$-g(x) \leqslant 0 \Rightarrow g^0(x;\eta(y,x)) \leqslant 0.$$

DEFINITION 6. (f,g) is said to be *pseudo type I* at $x \in C$, with respect to C, if for every $y \in C$, there is $\eta(y,x) \in T_C(x)$ such that

$$f^0(x;\eta(y,x)) \geqslant 0 \Rightarrow f(y) \geqslant f(x);$$
$$g^0(x;\eta(y,x)) \geqslant 0 \Rightarrow -g(x) \geqslant 0.$$

DEFINITION 7. (f,g) is said to be *quasipseudo type I* at $x \in C$, with respect to C, if for every $y \in C$, there is $\eta(y,x) \in T_C(x)$ such that

$$f(y) \leqslant f(x) \Rightarrow f^0(x;\eta(y,x)) \leqslant 0;$$
$$g^0(x;\eta(y,x)) \geqslant 0 \Rightarrow -g(x) \geqslant 0.$$

If in the above definition, we have

$$g^0(x;\eta(y,x)) \geqslant 0 \Rightarrow -g(x) > 0,$$

then, we say that (f,g) is *quasistrictlypseudo type I* at $x \in C$.

DEFINITION 8. (f,g) is said to be *pseudoquasi type I* at $x \in C$, with respect to C, if for every $y \in C$, there is $\eta(y,x) \in T_C(x)$ such that

$$f^0(x;\eta(y,x)) \geqslant 0 \Rightarrow f(y) \geqslant f(x);$$
$$-g(x) \leqslant 0 \Rightarrow g^0(x;\eta(y,x)) \leqslant 0.$$

We use the notion of generalized invexity (type I, pseudo type I, quasi type I, etc.) for functions between Banach spaces in a broad sense. Formally, in the following sense, we say $f\colon E \to F$ and $g\colon E \to G$ are type I, quasitype I, pseudotype I, quasipseudo-type I, pseudoquasi type I at $x \in C$ if $u^* \circ f$ and $v^* \circ g$ are type I, quasitype-I, pseudo-type I, quasi-pseudotype I, pseudo quasi type I, in the sense of Definitions 4, 5, 6, 7 and 8, respectively, for all $u^* \in Q^*$ and $v^* \in K^*$.

3. Optimality Conditions

Consider the generalized Pareto optimization problem defined by

$$\min\{f(x)\colon x \in C, -g(x) \in K\}, \tag{P}$$

where $f\colon E \to F$ and $g\colon E \to G$ are strongly compact Lipschitzian at $x_0 \in E$, $K \subset G$ is a pointed closed convex cone with nonempty interior, and C is a nonempty subset of E.

Let $\Im$ denote the set of all feasible solutions of problem (P), assumed to be nonempty, that is,

$$\Im = \{x \in C\colon g(x) \leqslant 0\} \neq \phi.$$

DEFINITION 9. We say that $x_0 \in \Im$ is a *weak Pareto-optimal solution* of problem (P), if there exists no $x \in \Im$ such that $f(x) < f(x_0)$.

The following proposition is from (Ref. [12]).

PROPOSITION 1. *If $x_0 \in \Im$ is a weak Pareto-optimal point for* (P), *then there exists a nonzero pair of vectors $(u^*, v^*) \in Q^* \times K^*$ such that, for some $k > 0$,*

$$0 \in \partial(u^* \circ f + v^* \circ g + k\delta_C)(x_0);$$
$$\langle v^*, g(x_0)\rangle = 0.$$

We adopt the following Slater-type constraint qualification:

DEFINITION 10. We say the restrictions of problem (P) satisfy the Slater condition if there exists $\hat{x} \in C$ such that $g(\hat{x}) < 0$.

In the rest of this section, we suppose that the restriction of (P) satisfy the Slater condition.

THEOREM 1 (Sufficient optimality). *Suppose that there exist $x_0 \in \Im$ and $u^* \in Q^*$, $u^* \neq 0$, $v^* \in K^*$, such that, for some $k > 0$,*

$$0 \in \partial(u^* \circ f + v^* \circ g + k\delta_C)(x_0); \tag{3}$$
$$\langle v^*,\ g(x_0)\rangle = 0. \tag{4}$$

If $(u^* \circ f, v^* \circ g)$ are type-I at $x_0 \in \Im$, with respect to C, then x_0 is a weak Pareto-optimal solution of (P).

Proof. Suppose that x_0 is not a weak Pareto-optimal solution of (P), then there exists $\hat{x} \in \Im$ such that $f(\hat{x}) - f(x_0) < 0$.

Since $u^* \neq 0$, we get

$$\langle u^*, f(\hat{x}) - f(x_0) \rangle < 0. \tag{5}$$

By the type-I hypothesis on f at x_0, there is $\eta(\hat{x}, x_0) \in T_C(x_0)$, such that

$$(u^* \circ f)^0(x_0; \eta(\hat{x}, x_0)) \leqslant \langle u^*, f(\hat{x}) - f(x_0) \rangle.$$

Combining this inequality with (5), we obtain

$$(u^* \circ f)^0(x_0; \eta(\hat{x}, x_0)) < 0. \tag{6}$$

Moreover, the type-I assumption on g at x_0 implies that, for the same, $\eta(\hat{x}, x_0) \in T_C(x_0)$, we have

$$(v^* \circ g)^0(x_0; \eta(\bar{x}, x_0)) \leqslant \langle v^*, -g(x_0) \rangle.$$

Since $\hat{x} \in \Im$ and (4) holds, we get

$$(v^* \circ g)^0(x_0; \eta(\hat{x}, x_0)) \leqslant 0. \tag{7}$$

From (6) and (7), we get

$$(u^* \circ f)^0(x_0; \eta(\hat{x}, x_0)) + (v^* \circ g)^0(x_0; \eta(\hat{x}, x_0)) < 0. \tag{8}$$

However, from (3), we get

$$0 \leqslant (u^* \circ f)^0(x_0; \eta) + (v^* \circ g)^0(x_0; \eta), \quad \forall \eta \in T_C(x_0),$$

which contradicts (8). Therefore, x_0 is a weak Pareto-optimal solution of (P).

THEOREM 2. *Suppose that there exist* $x_0 \in \Im$ *and* $u^* \in Q^*$, $u^* \neq 0$, $v^* \in K^*$ *such that, for some* $k > 0$, (3) *and* (4) *of Theorem* 3.1 *hold. If* (f, g) *are pseudoquasitype I at* x_0, *with respect to C, for the same* η, *then* x_0 *is a weak Pareto-optimal solution of* (P).

Proof. Since, $\langle v^*, \ g(x_0) \rangle = 0$ and (f, g) is pseudoquasi-type I at x_0, we have

$$(v^* \circ g)^0(x_0; \eta(\hat{x}, x_0)) \leqslant 0.$$

By using the above inequality in (3), we get

$$(u^* \circ f)^0(x_0; \eta(\hat{x}, x_0)) \geqslant 0 \quad \forall \eta \in T_C(x_0)$$
$$\Rightarrow \langle u^*, f(\hat{x}) - f(x_0) \rangle \geqslant 0$$
$$\Rightarrow f(\hat{x}) > f(x_0) (\text{because } u^* \neq 0).$$

Therefore, x_0 is a weak Pareto-optimal solution of (P).

The proof of the following theorem is easy and hence omitted.

THEOREM 3. *Suppose that there exist* $x_0 \in \Im$ *and* $u^* \in Q^*$, $u^* \neq 0$, $v^* \in K^*$ *such that, for some* $k > 0$, (3) *and* (4) *of Theorem* 1 *hold. If* (f, g) *are quasistriclty pseudo-type I at* x_0, *with respect C, for same* $\eta \in T_C(x_0)$, *then* x_0 *is a weak Pareto optimal solution of* (P).

4. Duality Results

We consider the following dual of problem (P):

$$\begin{aligned}&\max f(w), \qquad\qquad (D)\\ &\text{subject to}\\ &\quad w\in C, u^*\in Q^*, \quad u^*\neq 0, \quad v^*\in K^*,\\ &\quad \langle v^*, g(w)\geqslant 0, \quad 0\in\partial(u^*\circ f+v^*\circ g+k\delta_C)(w).\end{aligned}$$

In this section, we provide weak and strong duality relations between Problems (P) and (D).

THEOREM 4 (Weak duality). *Let x and (w,u^*,v^*) be feasible solutions for problems* (P) *and* (D), *respectively. Suppose that (f,g) are type-I at w with respect to C, for the same η. Then,*

$$f(x)<f(w).$$

Proof. Contrary to the result, suppose that there are feasible solutions $\hat{x}$ and (w,u^*,v^*) for problems (P) and (D), respectively, such that $f(\hat{x})-f(w)<0$. Since, $u^*\neq 0$, we obtain

$$\langle u^*, f(\hat{x})-f(w)\rangle<0.$$

By the first part of the assumption, there is $\eta(\hat{x},w)\in T_C(w)$ such that

$$(u^*\circ f)^0(w,\eta(\hat{x},w))\leqslant\langle u^*, f(\hat{x})-f(w)\rangle. \tag{9}$$

Hence, $(u^*\circ f)^0(w,\eta(\hat{x},w))<0$.
Since, $\langle v^*, g(w)\rangle\geqslant 0$, we get

$$-\langle v^*, g(w)\rangle\leqslant 0\Rightarrow(v^*\circ g)^0(w,\eta(\hat{x},w))\leqslant 0, \tag{10}$$

using second part of the hypothesis of type-I function.
Then, from (9) and (10), we have

$$(u^*\circ f)^0(w,\eta(\hat{x},w))+(v^*\circ g)^0(w,\eta(\hat{x},w))<0. \tag{11}$$

On the other hand, since $0\in\partial(u^*\circ f+v^*\circ g+k\delta_C)(w)$, we have

$$0\leqslant(u^*\circ f)^0(w;\eta)+(v^*\circ g)^0(w,\eta), \quad \forall\eta\in T_C(w),$$

which contradicts (11). Therefore, $f(x)<f(w)$.

THEOREM 5 (Weak duality). *Let x and (w,u^*,v^*) be feasible solutions for Problems* (P) *and* (D), *respectively. Suppose that (f,g) are pseudo-quasi-type I at w with respect to C, for the same η. Then, $f(x)<f(w)$.*

Proof. Contrary to the conclusion, suppose that there are feasible solutions $\hat{x}$ and (w, u^*, v^*) for problems (P) and (D), respectively such that $f(\hat{x}) - f(w) < 0$. Since $u^* \neq 0$, we obtain

$$\langle u^*, f(\hat{x}) - f(w) \rangle < 0.$$

By the first part of the assumption on f at w, there is $\eta(\hat{x}, w) \in T_C(w)$ such that

$$(u^* \circ f)^0(w, \eta(\hat{x}, w)) \leqslant \langle u^*, f(\hat{x}) - f(w) \rangle.$$

Hence,

$$(u^* \circ f)^0(w, \eta(\hat{x}, w)) < 0. \tag{12}$$

Since, $-\langle (v^*, g(w) \rangle \leqslant 0$, we have

$$(v^* \circ g)^0(w, \eta(\hat{x}, w)) \leqslant 0. \tag{13}$$

Adding (12) and (13), we get

$$(u^* \circ f)^0(w, \eta(\hat{x}, w)) + (v^* \circ g)^0(w, \eta(\hat{x}, w)) < 0. \tag{14}$$

On the other hand, since $0 \in \partial(u^* \circ f + v^* \circ g + k\delta_C)(w)$, we have

$$0 \leqslant (u^* \circ f)^0(w; \eta) + (v^* \circ g)^0(w, \eta), \quad \forall \eta \in T_C(w),$$

which contradicts (14). Therefore, $f(x) < f(w)$.

The proof of the following theorem is similar.

THEOREM 6 (Weak duality). *Let x and (w, u^*, v^*) be feasible solutions for Problems* (P) *and* (D), *respectively. Suppose that (f, g) are quasistrictly-pseudo-type I at w, with respect to C, for the same η. Then, $f(x) < f(w)$.*

THEOREM 7 (Strong duality). *Suppose that (f, g) are type I at all feasible points x of* (P), *with respect to C, and assume that the restrictions of Problem* (P) *satisfy the Slater condition. If x_0 is a weak Pareto-optimal solution of* (P), *then there exists $(\bar{u}^*, \bar{v}^*) \in Q^* \times K^*$ such that $\langle \bar{v}^*, g(x_0) \rangle = 0, (x_0, \bar{u}^*, \bar{v}^*)$ is a weak Pareto-optimal solution for* (D), *and the objective values of the two problems are the same.*

Proof. Since the Slater condition is satisfied, from Proposition 1 it follows that there exist $\bar{u}^*, \bar{v}^*$ such that $\langle \bar{v}^*, g(x_0) \rangle = 0$ and $(x_0, \bar{u}^*, \bar{v}^*)$ is feasible for (D). Suppose that $(x_0, \bar{u}^*, \bar{v}^*)$ is not an optimal solution for (D). So there exists a feasible point (x, u^*, v^*) for (D) such that $f(x) > f(x_0)$, which contradicts Theorem 4. Hence, $(x_0, \bar{u}^*, v^*)$ is a weak Pareto-optimal solution for (D). It is obvious that the objective function values of (P) and (D) are equal at their respective weak Pareto-optimal solutions.

THEOREM 8 (Strong duality). *Suppose that* (f, g) *are pseudo-quasi type I at all feasible points x of* (P), *with respect to C, and assume that the restrictions of* (P) *satisfy the Slater condition. If* x_0 *is a weak Pareto-optimal solution of* (P), *then there exists* $(\bar{u}^*, \bar{v}^*) \in Q^* \times K^*$ *such that* $\langle \bar{v}^*, g(x_0) \rangle = 0, (x_0, \bar{u}^*, \bar{v}^*)$ *is a weak Pareto-optimal solution for* (D), *and the objective values of the two problems are the same.*

Proof. The proof of this theorem is similar to that of Theorem 7, except that here we invoke the weak duality of Theorem 5.

In the proof of the following strong duality Theorem, we need the weak duality theorem (Theorem 6), rest of the proof is similar to the proof of Theorem 7.

THEOREM 9 (Strong duality). *Suppose that* (f, g) *is quasistrictly-pseudotype I at all feasible points x of* (P), *with respect to C, and assume that the restrictions of* (P) *satisfy the Slater condition. If* x_0 *is a weak Pareto-optimal solution of* (P), *then there exists* $(\bar{u}^*, \bar{v}^*) \in Q^* \times K^*$ *such that* $\langle \bar{v}^*, g(x_0) \rangle = 0, (x_0, \bar{u}^*, \bar{v}^*)$ *is a weak Pareto-optimal solution for* (D), *and the objective values of the two problems are the same.*

5. Conclusions

This work provides global optimality conditions and duality results for a class of nonconvex vector optimization problems posed on Banach spaces. It is first shown that, under weaker invexity assumption (type-I, pseudo quasi type-I, quasistrictly pseudo type-I, etc.) on the objective and constraint mappings and under a constraint qualification, some Karush-Kuhn-Tucker type conditions are sufficient for optimality. Then, a nonsmooth dual is considered and various duality theorems are established.

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